Bilinear Transformation
or
Mobius Transformation.

① Bilinear Transformation.

formula.

$$\boxed{w = \frac{az+b}{cz+d}}$$

&

$$\boxed{z = \frac{-dw+b}{cw-a}}$$

② Invariant Points of Bilinear Transformation

$$z = \frac{az+b}{cz+d}.$$

(3) Cross-Ration

if there are four Points z_1, z_2, z_3, z_4 taken in order then the ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$ is called the Cross-ration of z_1, z_2, z_3, z_4

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x

Method to find Bilinear Transformation.

(1) By finding a, b, c, d for $w = \frac{az+b}{cz+d}$ with the given Conditions

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(2) if three Point z_1, z_2, z_3 of the z-Plane & w_1, w_2, w_3 of the w-Plane

$$\frac{[w - w_1][w_2 - w_3]}{[w - w_3][w_2 - w_1]} = \frac{[z - z_1][z_2 - z_3]}{[z - z_3][z_2 - z_1]}$$

Example - 1

Find the bilinear Transformation which maps the Points $z = 1, i, -1$ into the Points $w = i, 0, -i$

Solution.

z	w
1	i
i	0
-1	$-i$

$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

$$w = \frac{\frac{a}{d}z + \frac{b}{d}}{\frac{c}{d}z + 1}$$

Now $\frac{a}{d} = p$, $\frac{b}{d} = q$, $\frac{c}{d} = r$

$$w = \frac{pz+q}{rz+1} \quad \text{--- (2)}$$

Now using Point z & w and find p, q, r

6 - (3)

(1) $z = 1, \quad w = i$

$$w = \frac{pz + q}{rz + 1}$$

$$i = \frac{p + q}{r + 1}$$

$$p + q = ir + i \quad \text{——— (A)}$$

(2) $z = i, \quad w = 0$

$$w = \frac{pz + q}{rz + 1}$$

$$0 = \frac{ip + q}{ir + 1}$$

$$pi + q = 0 \quad \text{——— (B)}$$

(3) $z = -1, \quad w = -i$

$$w = \frac{pz + q}{rz + 1}$$

$$-i = \frac{-p + q}{-r + 1}$$

$$-p + q = ir - i \quad \text{——— (C)}$$

6 — (4)

$p+q = ir+i$ —— (A)

$pi+q=0$ —— (B)

$-p+q = ir-i$ —— (C)

on Subtracting (C) from (A)

we get $2p = 2i$

$$\boxed{p = i}$$

on Putting the value of p in (B)

we have $i(i)+q=0$

$$\boxed{q = 1}$$

on substituting the values of p and q

in (A)

$$i+1 = ir+i$$

$$1 = ir$$

$$\boxed{r = -i}$$

Putting the values of p, q, r in eqn (2)

$$w = \frac{iz+1}{-iz+1}$$

or

Second Method Example — 1

Solution

z	w
1	i
i	0
-1	$-i$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)(\cancel{i})}{(w+i)(-\cancel{i})} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)}{(w+i)} = -\frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)}{(w+i)} = \frac{(1-z)(i+1)}{(z+1)(i-1)}$$

$$(w-i)\left[(z+1)(i-1)\right] = (w+i)\left[(1-z)(i+1)\right]$$

$$(w-i)\left[zi+i-z-1\right] = (w+i)\left[i-zi+1-z\right]$$

$$wzi+wi-wz-w-zi^2-i^2+iz+i$$

$$= wi-wzi+w-wz+i^2-zi^2+i-iz$$

$$wzi+\cancel{wi}-\cancel{wz}-w+\cancel{z}+1+iz+\cancel{i}-\cancel{wi}+wzi$$

$$-w+\cancel{wz}+1-\cancel{z}-\cancel{i}+iz=0$$

$$2wzi-2w+2iz+2=0$$

$$wzi-w+iz+1=0$$

$$w(zi-1) = -iz-1$$

$$\cancel{-}w\left[-zi+1\right] = \cancel{-}(iz+1)$$

$$\boxed{w = \frac{iz+1}{-iz+1}}$$

Example – 2

Find the bilinear Transformation which maps the Points $z = 0, -1, i$ onto $w = i, 0, \infty$

Solution.

z	w
0	i
-1	0
i	∞

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{\cancel{w_3}\,(w-w_1)\left[\frac{w_2}{w_3}-1\right]}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left[\frac{w_2}{w_3}-1\right]}{\left(\frac{w}{w_3}-1\right)\left[w_2-w_1\right]} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)\left[\frac{w_2}{\infty}-1\right]}{\left(\frac{w}{\infty}-1\right)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\frac{(w-i)\cancel{(-1)}}{(-i)\cancel{(-1)}} = +\frac{z(1+i)}{(z-i)\cancel{(-1)}}$$

$$\frac{(w-i)}{\cancel{-}i} = \frac{z(1+i)}{\cancel{-}(i-z)}$$

$$\frac{w-i}{i} = \frac{z(1+i)}{(i-z)}$$

$$(w-i)(i-z) = zi(1+i)$$

$$wi - wz - i^2 + iz = zi + zi^2$$

$$wi - wz + 1 + iz = zi - z$$

$$wi - wz + 1 + \cancel{iz} - \cancel{zi} + z = 0$$

$$wi - wz + 1 + z = 0$$

$$w(i-z) = -(1+z)$$

$$w = \frac{-(1+z)}{(i-z)} \Rightarrow w = \frac{(1+z)}{(z-i)}$$

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