Cauchy's Residue Theorem

if $f(z)$ = is analytic in a Closed curve C, except at a finite number of Poles within C, then $\int_C f(z)\,dz = 2\pi i\left[\begin{array}{l}\text{Sum of} \\ \text{Residues at} \\ \text{the Poles} \\ \text{within } C\end{array}\right]$

$$\int_C f(z)\,dz = 2\pi i\left[\begin{array}{l}\text{Sum of Residues at the Poles} \\ \text{with in } C.\end{array}\right]$$

Question ①

Evaluate the following integral using Residue theorem $\int_C \frac{1+z}{z(2-z)}\,dz$ where C is the circle $|z|=1$.

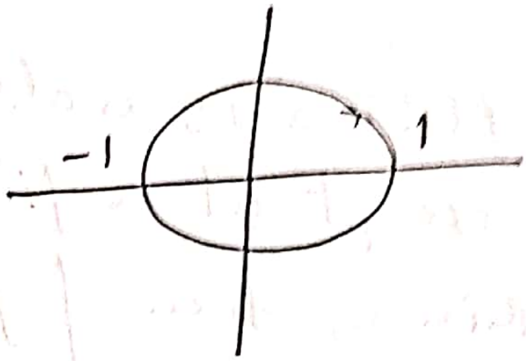
Solution: $\int_C \frac{1+z}{z(2-z)}\,dz$.

The Poles of the integrand are given by Putting the denominator equal to zero

$z(2-z) = 0$ $\qquad\qquad$ $|z| = 1$

$z = 0, 2$

As a Pole at $z=0$ inside the circle $|z|=1$



&

As a Pole at $z=2$ outside the circle $|z|=1$

Now find the Residue at the Pole $z=0$

$$\text{Res}(\text{at } z=z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$\text{Res}(\text{at } z=0) = \lim_{z \to 0} \cancel{(z-0)} \left[\frac{1+z}{\cancel{z}(2-z)} \right]$$

$$= \lim_{z \to 0} \left(\frac{1+z}{2-z} \right)$$

$$= \frac{1+0}{2-0} = \frac{1}{2}$$

using Cauchy's Residue theorem

$$\int f(z)\,dz = 2\pi i \left[\text{Sum of Residue at the Poles within } C\right]$$

$$\int \frac{1+z}{z(2-z)}\,dz = 2\pi i \left[\text{Res at } z=z_0\right]$$

$$= \cancel{2}\pi i \left(\frac{1}{\cancel{2}}\right)$$

$$= \pi i$$

$$\boxed{\int \frac{1+z}{z(2-z)}\,dz = \pi i}$$

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Question 2

Evaluate the following integral using Residue Theorem

$$\int_C \frac{z^2}{(z-1)(z-2)(z-3)}\, dz$$

where C is the circle $|z| = 4$.

Solution

$$\int_C \frac{z^2}{(z-1)(z-2)(z-3)}\, dz$$

The Poles of the integrand are given by Putting the denominator equal to zero

$$(z-1)(z-2)(z-3) = 0$$

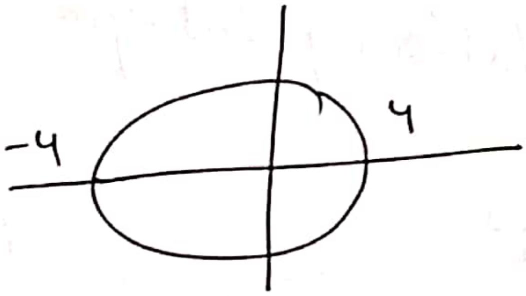
$$z = 1, 2, 3$$

The integrand is analytic on $|z| = 4$ and all Points inside

All a Pole at $z = 1, 2, 3$ inside the circle $|z| = 4$

R-3

$|z| = 4$



at the Pole $z = 1$

$$\text{Res}\,(\text{at } z = z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

$$\text{Res}\,(\text{at } z = 1) = \lim_{z \to 1} \cancel{(z-1)} \left[\frac{z^2}{\cancel{(z-1)}(z-2)(z-3)}\right]$$

$$= \lim_{z \to 1} \left[\frac{z^2}{(z-2)(z-3)}\right]$$

$$= \left[\frac{(1)^2}{(1-2)(1-3)}\right] = \frac{1}{2}$$

at the Pole $z = 2$

$$\text{Res}\,(\text{at } z = z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

$$\text{Res}\,(\text{at } z = 2) = \lim_{z \to 2} \cancel{(z-2)} \left[\frac{z^2}{(z-1)\cancel{(z-2)}(z-3)}\right]$$

R-4

$$= \lim_{z \to 2} \left[\frac{z^2}{(z-1)(z-3)} \right]$$

$$= -4$$

at the Poles $z = 3$

$$\text{Res}\,(\text{at } z = z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

$$\text{Res}\,(\text{at } z = 3) = \lim_{z \to 3} \cancel{(z-3)} \left[\frac{z^2}{(z-1)(z-2)\cancel{(z-3)}} \right]$$

$$= \lim_{z \to 3} \left[\frac{z^2}{(z-1)(z-2)} \right]$$

$$= \frac{(3)^2}{(3-1)(3-2)} = \frac{9}{2}$$

Now Cauchy's Residue theorem

$$\int_c f(z)\, dz = 2\pi i \left[\text{Sum of Residues at the Poles within } c \right]$$

$$= 2\pi i \left[\frac{1}{2} - 4 + \frac{9}{2} \right]$$

$$= 2\pi i \left[\frac{1 - 8 + 9}{2} \right] = 2\pi i \left(\frac{\cancel{2}}{\cancel{2}} \right)$$

$$\int_c \frac{z^2}{(z-1)(z-2)(z-3)}\, dz = 2\pi i$$

R-5

Question 3

Using Residue theorem evaluate $\int_c \frac{z^2}{(z-1)^2(z-3)} dz$
where c is the circle $|z|=2$

Solution $\int_c \frac{z^2}{(z-1)^2(z-3)} dz$

$(z-1)^2(z-3)=0$

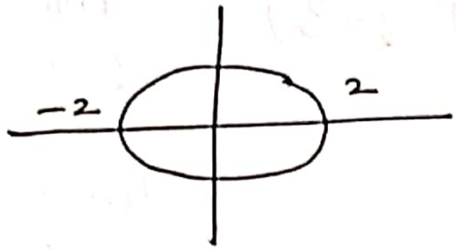
$z=1, 1, 3$

Now at the Pole
$z=3$ out side

&

$z=1$ inside and order n

Now $|z|=2$



at the Pole $z=1$ & $n=2$

$$Res(at\, z=z_0) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z-z_0)^n f(z)\right\}\right]_{z=z_0}$$

$$Res(at\, z=1) = \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}\left\{\cancel{(z-1)^2}\left(\frac{z^2}{\cancel{(z-1)^2}(z-3)}\right)\right\}\right]_{z=1}$$

$$= \left[\frac{d}{dz}\left(\frac{z^2}{z-3}\right)\right]_{z=1}$$

R-6

$$\text{Res (at } z=1) = \left[\frac{(z-3)\frac{d}{dz}z^2 - z^2\frac{d}{dz}(z-3)}{(z-3)^2}\right]_{z=1}$$

$$= \left[\frac{2z(z-3) - z^2}{(z-3)^2}\right]_{z=1}$$

$$= \frac{2(1-3) - 1}{(1-3)^2} = \frac{-4-1}{(-2)^2}$$

$$= -\frac{5}{4}$$

Now by Residue Theorem

$$\int_c f(z)\,dz = 2\pi i \left[\begin{array}{c}\text{Sum of Residues at} \\ \text{the Pole within } c\end{array}\right]$$

$$= 2\pi i\left[-\frac{5}{4}\right]$$

$$= -\frac{5\pi i}{2}$$

$$\int_c \frac{z^2}{(z-1)^2(z-3)}\,dz = -\frac{5\pi i}{2}$$

R-7

Question (4)

Using Residue theorem evaluate

$$\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz \quad \text{where } C: |z| = 3$$

Solution $\displaystyle\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz$

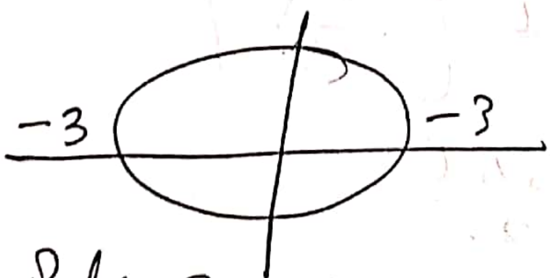
Pole of $f(z)$ are given by

$$(z-1)^2(z+2) = 0$$

$$z = 1, 1, -2$$

The Pole at $z=1$ is of second order and the Pole at $z=-2$ is simple

$$|z| = 3$$



at the Pole $z=-2$

$$\text{Res}(\text{at } z = z_0) = \lim_{z\to z_0} (z-z_0)\,f(z)$$

$$\text{Res}(\text{at } z=2) = \lim_{z\to -2} \cancel{(z-2)}\left[\frac{z^2}{\cancel{(z-2)}(z-1)^2}\right]$$

$$\text{Res (at } z=-2) = \lim_{z \to \cancel{z}-2} \left(\frac{z^2}{(z-1)^2} \right]$$

$$= \frac{(2)^2}{(-2-1)^2}$$

$$= \frac{4}{(3)^2}$$

$$= 4/9$$

at the Pole $z=1$

$$\text{Res (at } z=z_0) = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} \left\{ (z-z_0)^n f(z) \right\} \right]_{z=z_0}$$

$$= \frac{1}{(2-1)!} \left[\frac{d^{2-1}}{dz^{2-1}} \left\{ \cancel{(z-1)^2} \left\{ \frac{z^2}{\cancel{(z-1)^2}(z+2)} \right\} \right\} \right]_{z=1}$$

$$= \left[\frac{d}{dz} \left(\frac{z^2}{z+2} \right) \right]_{z=1}$$

$$= \left[\frac{(z+2) \frac{d}{dz} z^2 - z^2 \frac{d}{dz}(z+2)}{(z+2)^2} \right]_{z=1}$$

$$= \left[\frac{2z(z+2) - z^2}{(z+2)^2} \right]_{z=1}$$

R-9

$$\text{Res}\,(\text{at } z=1) = \frac{2(1+2)-1}{(1+2)^2}$$

$$= \frac{6-1}{(3)^2}$$

$$= \frac{5}{9}$$

$$\int f(z)\,dz = 2\pi i\left[\text{Sum of Res at the Pole with in } C\right]$$

$$= 2\pi i\left[\frac{4}{9} + \frac{5}{9}\right]$$

$$= 2\pi i$$