3^{***}

Conjugate Function.

If $f(z) = u + iv$

Case – I –: if $u = u(x,y)$ is given

if Given and u is Known

to find v conjugate function.

$$\boxed{dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy}$$

or

$$\boxed{dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.}$$

Case – II

if $V = V(x, y)$ is given

and v is known

To find u, ~~conjug~~ Conjugate function

$$\boxed{du = \frac{\partial u}{\partial x}\,dx + i\,\frac{\partial u}{\partial y}\,dy}$$

or

$$\boxed{dU = \frac{\partial U}{\partial x}\,dx + i\,\frac{\partial U}{\partial y}\,dy}$$

Example - 1

Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. find a function v such that $f(z) = u + iv$ is analytic. and find $f(z) = ?$

Solution:. $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2$$

$$\frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$\frac{\partial^2 u}{\partial y^2} = -2$$

using Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$2 - 2 = 0$$

$$0 = 0$$

Since Laplace equation is satisfied therefore u is harmonic.

Now then find v conjugate function.

we know that

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \text{—— (A)}$$

Now using C-R equation

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Putting in eqⁿ (A)

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{or} \quad \boxed{\partial v = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy}$$

$$dv = -(-2y - 2x + 3)\, dx + (2x - 2y - 2)\, dy$$

$$dv = (2y + 2x - 3)\, dx + (2x - 2y - 2)\, dy$$

$$dv = 2y\,dx + 2x\,dx - 3\,dx + 2x\,dy - 2y\,dy - 2\,dy$$

$$dv = [2y\,dx + 2x\,dy] + 2x\,dx - 3\,dx - 2y\,dy - 2\,dy$$

$$dv = 2d(xy) + d(x^2) - 3d(x) - d(y^2) - 2d(y)$$

both side use integral.

$$\int dv = 2\int d(xy) + \int d(x^2) - 3\int d(x) - \int d(y^2) - 2\int d(y)$$

$$v = 2xy + x^2 - 3x - y^2 - 2y$$

$$f(z) = u + iv$$

$$f(z) = [x^2 - y^2 - 2xy - 2x + 3y] + i[2xy + x^2 - 3x - y^2 - 2y]$$

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Example – 2

Find the imaginary Part of the analytic function and find $f(3)$, whose real Part is $x^3 - 3xy^2 + 3x^2 - 3y^2$

Solution.

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

we know that

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \text{——— (A)}$$

Now using C-R equation

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Putting in eqn (A)

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$dv = -(-6xy-6y)\,dx + (3x^2-3y^2+6x)\,dy$$

$$dv = (6xy+6y)\,dx + (3x^2-3y^2+6x)\,dy$$

$$dv = 6xy\,dx + 6y\,dx + 3x^2dy - 3y^2dy + 6x\,dy$$

$$dv = [6xy\,dx + 3x^2dy] + [6y\,dx + 6x\,dy] - 3y^2dy$$

$$dv = 3d(x^2y) + 6d(xy) - d(y^3)$$

$$\int dv = 3\int d(x^2y) + 6\int d(xy) - \int d(y^3)$$

$$v = 3x^2y + 6xy - y^3$$

$$f(z) = u + iv$$

$$= [x^3 - 3xy^2 + 3x^2 - 3y^2] + i[3x^2y + 6xy - y^3]$$