Residues

Method of Finding Residues

(a) Residues at simple Pole

$$\text{Res}(\text{at } z = z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

where z_0 = Poles.

(b) Residues at a Pole of order n.

$$\text{Res}(\text{at } z = z_0) = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \right\}_{z = z_0}$$

where z_0 is Poles.

n is How many Pole of order.

Question ①

Determine the Pole and residue at the Pole of the function $f(z) = \frac{z}{z-1}$

Solution :

$$f(z) = \frac{z}{z-1}$$

The Poles is determined by Putting the denominator equal to zero

$$z-1=0$$

$$z=1$$

These Poles is only one $z=1$

at the Pole $z=1$ using simple Pole

$$Res\,(at\ z=z_0) = \lim_{z \to z_0} (z-z_0)\, f(z)$$

$$Res\,(at\ z=1) = \lim_{z \to 1} \cancel{(z-1)} \left[\frac{z}{\cancel{(z-1)}}\right]$$

$$= \lim_{z \to 1} (z)$$

$$= 1.$$

Question (2) Find the Residue of a function

$$f(z) = \frac{z^2}{(z-4)(z-5)}$$

Solution:

$$f(z) = \frac{z^2}{(z-4)(z-5)}$$

$$(z-4)(z-5) = 0$$

$$z = 4, 5$$

at the Pole $z = 4$

$$\text{Res}\,(\text{at } z = z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

$$\text{Res}\,(\text{at } z = 4) = \lim_{z \to 4} \cancel{(z-4)} \left[\frac{z^2}{\cancel{(z-4)}(z-5)} \right]$$

$$= \lim_{z \to 4} \frac{z^2}{(z-5)}$$

$$= \frac{(4)^2}{4-5}$$

$$= \frac{16}{-1}$$

$$= -16$$

at the Pole $z = 5$

$$\text{Res}(\text{at } z = z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

$$\text{Res}(\text{at } z = 5) = \lim_{z \to 5} \cancel{(z-5)} \left[\frac{z^2}{(z-4)\cancel{(z-5)}} \right]$$

$$= \lim_{z \to 5} \frac{z^2}{z-4}$$

$$= \frac{(5)^2}{5-4}$$

$$= 25$$

Question 3

Find the Residue of a function

$$f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$$ & Sum.

Solution: $$f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$$

The Poles is determined by Putting the denominator equal to zero

$$(z-1)(z-2)(z-3) = 0$$

$$z = 1, 2, 3$$

at the Pole $z=1$

$$\text{Res}(\text{at } z=z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$\text{Res}(\text{at } z=1) = \lim_{z \to 1} \cancel{(z-1)} \left[\frac{z^3}{\cancel{(z-1)}(z-2)(z-3)} \right]$$

$$= \lim_{z \to 1} \left[\frac{z^3}{(z-2)(z-3)} \right]$$

$$= \frac{(1)^3}{(1-2)(1-3)}$$

$$= \frac{1}{(-1)(-2)}$$

$$= \frac{1}{2}$$

at the Pole $z=2$

$$\text{Res}(\text{at } z=z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$\text{Res}(\text{at } z=2) = \lim_{z \to 2} \cancel{(z-2)} \left[\frac{z^3}{(z-1)\cancel{(z-2)}(z-3)} \right]$$

$$= \lim_{z \to 2} \frac{z^3}{(z-1)(z-3)}$$

$$= \frac{(2)^3}{(2-1)(2-3)}$$

$$= \frac{8}{(1)(-1)}$$

$$\text{Res}(\text{at } z=2) = -8$$

at the Pole $z=3$

$$\text{Res}(\text{at } z=z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$\text{Res}(\text{at } z=3) = \lim_{z \to 3} \cancel{(z-3)} \left[\frac{z^3}{(z-1)(z-2)\cancel{(z-3)}} \right]$$

$$= \lim_{z \to 3} \frac{z^3}{(z-1)(z-2)}$$

$$= \frac{(3)^3}{(3-1)(3-2)}$$

$$= \frac{27}{(2)(1)}$$

$$= \frac{27}{2}$$

$$f(z) = \text{Res}(\text{at } z=1) + \text{Res}(\text{at } z=2) + \text{Res}(\text{at } z=3)$$

$$\frac{z}{(z-1)(z-4)} = \frac{1}{2} - 8 + \frac{27}{2}$$

$$= \frac{1-16+27}{2}$$

$$= \frac{28-16}{2}$$

$$= \frac{12}{2}$$

$$= 6.$$

(8)

Question
1

Find the Residue of a function $f(z) = \frac{z^2}{(z+1)^2}$

Solution:
1

$f(z) = \frac{z^2}{(z+1)^2}$

Poles are determined by putting denominator equal to zero.

$$(z+1)^2 = 0 \quad \Rightarrow (z+1)(z+1) = 0$$

$z = -1, -1$

The function has a double Pole at $z = -1$

$$\text{Residue (at } z = z_0) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z-z_0)^n f(z)\right\}\right]_{z=z_0}$$

$$\text{Residue (at } z = -1) = \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}\left\{(z+1)^2\left[\frac{z^2}{(z+1)^2}\right]\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz}\left\{\cancel{(z+1)^2}\left[\frac{z^2}{\cancel{(z+1)^2}}\right]\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz} z^2\right]_{z=-1}$$

$$= \left[2z\right]_{z=-1}$$

$$= -2$$

Question

Find the Residue of $\frac{1}{(z^2+1)^3}$ at $z=i$

Solution: $f(z) = \frac{1}{(z^2+1)^3}$

The Poles of $f(z)$ are determined by Putting denominator equal to zero.

$$(z^2+1)^3 = 0$$

$$(z+i)^3(z-i)^3 = 0$$

$$z = i, i, i, -i, -i, -i$$

Here $z=i$ is a Pole of order 3 of $f(z)$

$$\text{Res}(\text{at } z=z_0) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z-z_0)^n\left\{f(z)\right\}\right\}\right]_{z=z_0}$$

$$\text{Res}(\text{at } z=i) = \frac{1}{(3-1)!}\left[\frac{d^{3-1}}{dz^{3-1}}\left\{\cancel{(z-i)^3}\left\{\frac{1}{(z+i)^3\cancel{(z-i)^3}}\right\}\right\}\right]_{z=i}$$

$$= \frac{1}{2!}\left[\frac{d^2}{dz^2}\left\{\frac{1}{(z+i)^3}\right\}\right]_{z=i}$$

$$= \frac{1}{2}\left[\frac{d}{dz}\left\{\frac{d}{dz}\left(\frac{1}{z+i}\right)^3\right]_{z=i}$$

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$$\operatorname{Res}(\text{at } z=i) = \frac{1}{2}\left[\frac{d}{dz}\left\{\frac{-3}{(z+i)^4}\right\}\right]_{z=i}$$

$$= \frac{1}{\cancel{2}}\left[\frac{(-3)(-\cancel{4}^{2})}{(z+i)^5}\right]_{z=i}$$

$$= \left[\frac{6}{(z+i)^5}\right]_{z=i}$$

$$= \left[\frac{6}{(i+i)^5}\right]$$

$$= \frac{6}{(2i)^5}$$

$$= \frac{\cancel{6}}{3\cancel{2}(i)^5}$$

$$= \frac{3}{16i}$$

$$= \frac{-3i}{16}$$