Integration Round Unit circle of the Type

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta.$$

Introduction

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Convert $\sin\theta$, $\cos\theta$ into z

Consider a circle of unit Radius with centre at origin as Contour.

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{—— (1)}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad \text{—— (2)}$$

eq^n (1) & (2) $\Rightarrow$ (1) + (2)

$$2\cos\theta = e^{i\theta} + e^{-i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{—— (A)}$$

$eq^n ① - eq^n ②$

$$2i\sin\theta = e^{i\theta} - e^{-i\theta}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{———} \quad (B)$$

$$z = x + iy$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = re^{i\theta}$$

$$z = e^{i\theta}$$

$\because |z| = 1$

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}d\theta$$

$$dz = zi\,d\theta$$

$$d\theta = \frac{dz}{zi}$$

① $z = re^{i\theta}$ $\because |z|=1$

$z = e^{i\theta}$,

② $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sin\theta = \frac{z - z^{-1}}{2i}$$

$$= \frac{z^2 - 1}{2iz}$$

③ $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cos\theta = \frac{z + z^{-1}}{2}$$

$$\cos\theta = \frac{z^2 + 1}{2z}$$

P — ②

Question (1)

Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$

Solution :: $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$

We know

$$z = re^{i\theta} \qquad \because r=1$$

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}\, d\theta$$

$$dz = iz\, d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\cos\theta = \frac{z+z^{-1}}{2}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

Now

C is the Unit circle $|z|=1$

then 0 to $2\pi \Rightarrow |z|=1$

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5-3\left[\frac{e^{i\theta}+e^{-i\theta}}{2}\right]}$$

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{2\,d\theta}{10-3\left[e^{i\theta}+e^{-i\theta}\right]}$$

$$= \int_c \frac{2}{10-3\left[z+z^{-1}\right]} \cdot \left\{\frac{dz}{iz}\right\}$$

$$= \int_c \frac{2}{10-3\left[z+\frac{1}{z}\right]} \cdot \left\{\frac{dz}{iz}\right\}$$

$$= \int_c \frac{2z}{10z-3(z^2+1)} \cdot \left(\frac{dz}{iz}\right)$$

$$= \int_c \frac{2\cancel{z}}{-3z^2-3+10z} \cdot \left(\frac{dz}{i\cancel{z}}\right)$$

$$= -\frac{2}{i}\int_c \frac{dz}{3z^2-10z+3}$$

$$= -\frac{2i}{i^2}\int_c \frac{dz}{(3z-1)(z-3)}$$

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$$= 2i \int_C \frac{dz}{(3z-1)(z-3)}$$

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = 2i\, I \quad \text{——— (A)}$$

where C is unit circle $|z|=1$

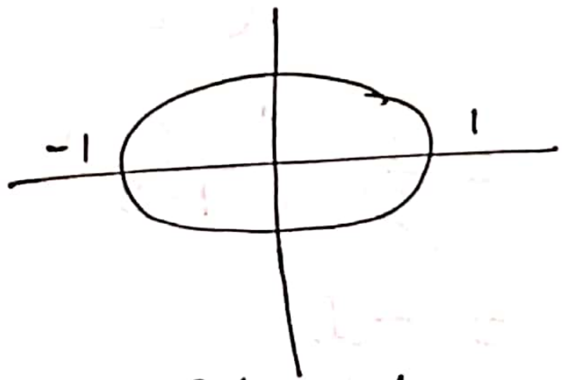
$$I = \int_C \frac{dz}{(3z-1)(z-3)}$$

Poles of integrand are given by

$$(3z-1)(z-3)=0$$

$$z = \frac{1}{3}, 3$$

$|z|=1$



There is only one Pole at $z=\frac{1}{3}$ inside the unit circle C.

Find Residue at the Pole $z=\frac{1}{3}$

$$\text{Res}\,[\text{at } z = z_0] = \lim_{z \to z_0} (z - z_0)\, f(z)$$

$$\text{Res}\,\left[\text{at } z = \tfrac{1}{3}\right] = \lim_{z \to \frac{1}{3}} \left(z - \tfrac{1}{3}\right) \left[\frac{1}{(3z-1)(z-3)}\right]$$

$$= \frac{1}{3} \lim_{z \to \frac{1}{3}} \cancel{(3z-1)} \left[\frac{1}{\cancel{(3z-1)}(z-3)}\right]$$

$$= \frac{1}{3} \lim_{z \to \frac{1}{3}} \left(\frac{1}{z-3}\right)$$

$$= \frac{1}{3} \left(\frac{1}{\frac{1}{3} - 3}\right)$$

$$= \frac{1}{\cancel{3}} \left(\frac{\cancel{3}}{1-9}\right)$$

$$= -\frac{1}{8}$$

Hence by Cauchy's Residue theorem

$$I = 2\pi i \left[\text{Sum of the Residues within Contour}\right]$$

$$I = 2\pi i\left[-\frac{1}{8}\right]$$

$$I = \cancel{2}\pi i\left(-\frac{1}{\cancel{8}}\right)$$

$$I = \frac{-\pi i}{4} \quad \text{——— (B)}$$

$$\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta} = 2i(I) \quad \text{——— (A)}$$

$$\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta} = \cancel{2}i\left[\frac{-\pi i}{\cancel{4}}\right]$$

$$= \frac{i\,[-\pi i]}{2}$$

$$= \frac{-\pi(i)^2}{2}$$

$$= \frac{\pi}{2}$$

$$\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta} = \frac{\pi}{2}$$

Ans//