Taylor's and Laurent's Series

Taylor's Theorem

if a function $f(z)$ is analytic at all Points inside a circle C, with its centre at the Point z_0 and radius R, then at each Point z inside C.

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + (z-z_0)^2 \frac{f''(z_0)}{2} + \cdots$$

$$\cdots + (z-z_0)^n \frac{f^n(z_0)}{n!} + \cdots$$

Laurent's Theorem

if $f(z)$ is analytic on c_1 and c_2 and the annular region R bounded by the two concentric circles c_1 and c_2 of radii r_1 and r_2 $(r_2 < r_1)$ and with centre at z_0 then for all z in R.

~~Concentric circles C_1 and C_2 of~~

where

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2}$$

$$+ \cdots \cdots$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \, dw$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} \, dw .$$

$$f(z) = \sum_{n=0}^{n=\infty} a_n (z-a)^n + \sum_{n=1}^{n=\infty} \frac{b_n}{(z-a)^n}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 \cdots\cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 \cdots\cdots$$

Question (1)

Obtain the Taylor or Laurent series which represents the function $f(z) = \frac{1}{(z-1)(z-2)}$ when

(i) $|z|<1$ (ii) $|z|>2$ (iii) $1<|z|<2$

Solution: $f(z) = \frac{1}{(z-1)(z-2)}$

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)} \quad \text{—— (A)}$$

(i) $|z|<1$

$$f(z) = \frac{1}{2}\left[\frac{1}{\frac{z}{2}-1}\right] - \frac{1}{(z-1)}$$

$$= \frac{-1}{2}\left[\frac{1}{1-\frac{z}{2}}\right] + \frac{1}{1-z}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$\cancel{f(z) = -\frac{1}{2}\left[1+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3 \cdots\right] + \left[1+z^2 \cdots\right.}$$

$$f(z) = -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\cdots\right] + \left[1+z+z^2+z^3+\cdots\cdots\right]$$

(ii) $|z|>2$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{z}\left[\frac{1}{\left(1-\frac{2}{z}\right)}\right] - \frac{1}{z}\left[\frac{1}{\left(1-\frac{1}{z}\right)}\right]$$

$$= \frac{1}{z}\left[\left(1-\frac{2}{z}\right)^{-1} - \left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{z}\left[\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\cdots\right) - \left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\cdots\right)\right]$$

(iii) $1<|z|<2$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= -\frac{1}{2}\left[\frac{1}{1-\frac{z}{2}}\right] - \frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right]$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\cdots\right]$$

Question (2)

Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent and Taylor's Series valid for.

(i) $1 < |z| < 3$ (ii) $|z| > 3$

(iii) $|z| < 1$ (iv) $0 < |z+1| < 2$

Solution :-

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$f(z) = \frac{1}{2}\left[\frac{1}{(z+1)} - \frac{1}{(z+3)}\right].$$

(i) $|z| < 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{(z+1)} - \frac{1}{(z+3)}\right]$$

$$f(z) = \frac{1}{2}\left[\frac{1}{(1+z)} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right]$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 \cdots\right]$$

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(ii) $|z| > 3$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{(z+3)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) - \frac{1}{z}\left(\frac{1}{1+\frac{3}{z}}\right)\right]$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 \cdots\cdots\right) - \left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2 + \cdots\cdots\right)\right]$$

(iii) $1 < |z| < 3$

$$f(z) = \frac{1}{2}\left[\frac{1}{(z+1)} - \frac{1}{(z+3)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{3}\frac{1}{\left(1+\frac{z}{3}\right)}\right]$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\cdots\right) - \frac{1}{6}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right)$$

(iv) $0 < |z+1| < 2$

$$f(z) = \frac{1}{2}\left[\frac{1}{(z+1)} - \frac{1}{(z+3)}\right]$$

$$f(z) = \frac{1}{2}\left[\frac{1}{(z+1)} - \frac{1}{(z+1)+2}\right]$$

$$= \frac{1}{2}\left[(z+1)^{-1} - \frac{1}{2}\left\{\frac{1}{1+\frac{(z+1)}{2}}\right\}\right]$$

$$= \frac{1}{2}\left[z+1\right]^{-1} - \frac{1}{4}\left[1+\frac{(z+1)}{2}\right]^{-1}$$

$$= \frac{1}{2}\left(1 - z + z^2 - z^3 \cdots\right) - \frac{1}{4}\left[1 - \frac{(z+1)}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \cdots\right]$$

Question (3)

Obtain the Taylor or Laurent series which represents the function.

$$f(z) = \frac{1}{(1+z^2)(z+2)} \quad \text{when}$$

(i) $1 < |z| < 2$ (ii) $|z| > 2$

Solution:

$$f(z) = \frac{1}{(z^2+1)(z+2)}$$

$$\frac{1}{(z^2+1)(z+2)} = \frac{-\frac{z}{5}+\frac{2}{5}}{(1+z^2)} + \frac{\frac{1}{5}}{(z+2)}$$

$$f(z) = \frac{1}{5}\left[\frac{1}{z+2} - \frac{(z-2)}{(1+z^2)}\right]$$

(i) $|z| > 2$

$$f(z) = \frac{1}{5}\left[\frac{1}{(z+2)} - \frac{(z-2)}{(z^2+1)}\right]$$

$$= \frac{1}{5}\left[\frac{1}{z}\left\{\frac{1}{1+\frac{2}{z}}\right\} - \frac{(z-2)}{z^2}\left\{\frac{1}{1+\frac{1}{z^2}}\right\}\right]$$

$$f(z) = \frac{1}{5}\left[\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} - \left(\frac{z-2}{z^2}\right)\left\{1+\frac{1}{z^2}\right\}^{-1}\right]$$

$$= \frac{1}{5}\left[\frac{1}{z}\left\{1-\frac{2}{z}+\left(\frac{2}{z}\right)^2 - \cdots\right\} - \left(\frac{z-2}{z^2}\right)\left\{1-\frac{1}{z^2} + \left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3 + \cdots\right\}\right]$$

(ii) $1 < |z| < 2$