2

Harmonic Function

Any function which satisfies the Laplace's equation is known as a harmonic function.

Theorem:- If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic function.

Now

$f(z) = u + iv$.

$f(z)$ is Analytic function

u = is Real Parts

$u = f(x, y)$

v = is Imaginary Part.

if u is give then find Harmonic function

$u = f(x, y)$

① $\frac{\partial u}{\partial x}$, ② $\frac{\partial u}{\partial y}$, ③ $\frac{\partial^2 u}{\partial x^2}$, ④ $\frac{\partial^2 u}{\partial y^2}$

Now

$$\left.\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right] \text{Laplace's equation}$$

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if v is given then find Harmonic function

$v = f(x, y)$

① $\frac{\partial v}{\partial x}$, ② $\frac{\partial v}{\partial y}$, ③ $\frac{\partial^2 v}{\partial x^2}$, ④ $\frac{\partial^2 v}{\partial y^2}$

Now

$$\left.\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \right] \text{Laplace's equation.}$$

Therefore both u and v are harmonic function

Ex - (1)

Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic function.

Solution:.

$$u = \frac{1}{2}\log(x^2+y^2) \quad \text{---} (1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \text{---} (2)$$

again

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{---} (3)$$

(2) - (3)

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{1}{x^2+y^2}\right]\frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{1}{\cancel{2}}\left[\frac{1}{x^2+y^2}\right]\cancel{2}y$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \quad \text{——— (4)}$$

again

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\frac{\partial}{\partial y}y - y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{——— (5)}$$

using laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(2) — 4

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2} = 0$$

$$\frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2} = 0$$

$$\frac{0}{(x^2+y^2)^2} = 0$$

$$0 = 0$$

L.H.S = R.H.S

Since laplace equation is satisfied therefore u is harmonic.

(2) - (5)

Example — 2

Prove that $v = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic.

$$v = x^2 - y^2 - 2xy - 2x + 3y$$

$$\frac{\partial v}{\partial x} = 2x - 2y - 2$$

$$\frac{\partial^2 v}{\partial x^2} = 2$$

$$\frac{\partial v}{\partial y} = -2y - 2x + 3$$

$$\frac{\partial^2 v}{\partial y^2} = -2$$

using Laplace equations

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$2 - 2 = 0$$

$$0 = 0$$

Since Laplace equation is satisfied therefore v is harmonic.