③ Test the analyticity of the function $w = \sin z$ and hence derive that

$$\frac{d}{dz}(\sin z) = \cos z$$

Solution.

$$w = \sin z \quad \text{—— ①}$$

$$w = f(z) \quad \text{or} \quad f(z) = u + iv$$

$f(z) = \sin z$

Now $z = x + iy$

$u + iv = \sin z$

$u + iv = \sin(x + iy)$

$u + iv = \sin x . \cos(iy) + \cos x \sin(iy)$ —— ②

Now

$\cos iy = \cosh y$

$\sin iy = i \sinh y$

$u + iv = \sin x . \cosh y + i \cos x . \sinh y$

$u = \sin x . \cosh y$

$v = \cos x . \sinh y$

Q-3-1

$u = \sin x . \cosh y$

$\frac{\partial u}{\partial x} = \cos x . \cosh y$

$\frac{d}{dy} \cosh y = \sinh y$

$\frac{\partial u}{\partial y} = \sin x . \sinh y$

$v = \cos x . \sinh y$

$\frac{d}{dy} \sinh y = \cosh y$

$\frac{\partial v}{\partial x} = -\sin x . \sinh y$

$\frac{\partial v}{\partial y} = \cos x \cosh y$

using C-R equation

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

① $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\cos x . \cosh y = \cos x \cosh y$

② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\sin x . \sinh y = -(-\sin x \sinh y)$

$\sin x . \sinh y = \sin x . \sinh y$

So C-R equation are satisfied and Partial derivatives are continous

f(z) or w is analytic function.

Q-3-2

Hence $\sin z$ is an analytic function

Now find $f'(z)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$= \cos x \cos(iy) - \sin x\,(i \sinh y)$$

$$= \cos x \cos(iy) - \sin x . \sin(iy)$$

$$= \cos(x+iy)$$

$$f'(z) = \cos z$$

Q-3-3